WHITTAKER MODULES FOR THE W-ALGEBRA $W(2,2)$

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Abstract

In this paper, Whittaker modules for the W-algebra $W(2,2)$ are studied. We obtain analogues to several results from the classical setting and the case of the Virasoro algebra, including a classification of simple Whittaker modules by central characters and composition series for general Whittaker modules.

Keywords: the W-algebra $W(2,2)$, Whittaker modules, Whittaker vectors.

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§1.Introduction

In this paper we investigate Whittaker modules for the W-algebra $W(2,2)$. Whittaker modules were first discovered for $\mathfrak{sl}_2(\mathbb{C})$ by Arnal and Pinzcon in [1]. Block showed, in [3] that the simple modules for $\mathfrak{sl}_2(\mathbb{C})$ consist of highest (lowest) weight modules, Whittaker modules and a third family obtained by localization. This illustrates the prominent role played by Whittaker modules. The algebra $W(2,2)$ is related to vertex operator algebras. In [14], $W(2,2)$ plays an important role in the classification of simple vertex operator algebras generated by two weight vectors.

Kostant defined Whittaker modules for an arbitrary finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$ in [7], and showed that these modules, up to isomorphism, are in bijective correspondence with ideals of the center $Z(\mathfrak{g})$. In particular, irreducible Whittaker modules correspond to maximal ideals of $Z(\mathfrak{g})$. In the quantum setting, Whittaker modules have been studied by Sevoystanov for $\mathcal{U}_h(\mathfrak{g})$ in [13] and by M. Ondrus for $\mathcal{U}_q(\mathfrak{sl}_2)$ in [11]. Recently Whittaker modules have also been studied by M. Ondrus and E. Wiesner for the Virasoro algebra in [12], X. Zhang and S. Tan for the Schrödinger-Virasoro algebra in [15], K. Christodoulopoulou for Heisenberg algebras in [4], and by G. Benkart and M. Ondrus for generalized Weyl algebras in [2].

The algebra $W(2,2)$ is an infinite-dimensional Lie algebra. Recently, there appeared some papers on the structures and representations for this algebra. In [8], [10] and [14], its irreducible weight modules, indecomposable modules and Verma modules were respectively studied. Its derivations, central extensions and automorphisms were determined in [6]. The Lie bialgebra and left symmetric algebra structures on the algebra were determined in [9] and [5].

Our main result is Theorem 4.2 in which we get a classification of irreducible Whittaker modules by central characters. This theorem follows easily from two results, one is Proposition 3.1 by which the Whittaker vectors in a universal Whttaker module are determined,

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and the other is Lemma 3.3 which says that every submodule of a universal Whittaker module contains a Whittaker vector. We use the concrete nature of the algebra $W(2,2)$ to obtain 3.1, but our proof for 3.3, different from [12], is more general and could be possibly extended to infinitely generated algebras. The paper is organized as follows. In section 2, we define Whittaker vectors and Whittaker modules for $W(2,2)$, and also construct a universal Whittaker module. Then the Whittaker vectors in a universal Whittaker module are examined in section 3 and the irreducible Whittaker modules are classified in section 4. In the last section we describe a decomposition of an arbitrary Whittaker module and characterize its submodules.

§2. Preliminaries

2.1. The algebra $W(2,2)$, denoted by $\mathcal{W}$, is an infinitely dimensional Lie algebra with a $\mathbb{C}$-basis $\{\, L_n,\, W_n,\, z \,|\, n \in \mathbb{Z}\,\}$ and the following Lie brackets:

$$[L_n, L_m\,] = (m-n)L_{m+n} + \tfrac{n^3-n}{12}\delta_{m+n,0}z,$$
$$[L_n, W_m] = (m-n)W_{m+n} + \tfrac{n^3-n}{12}\delta_{m+n,0}z.$$
$$[W_n, W_m] = [z, W_m] = [z, L_m] = 0$$

Let $S(z)$represent the symmetric algebra generated by z, that is, plolynomials in z. Then $S(z)$ is obviously contained in the center of the universal enveloping algebra $\mathcal{U}(\mathcal{W})$.

Set $\mathcal{W}_n = Span_{\mathbb{C}}\{L_n, W_n\}, n \neq 0, \mathcal{W}_0 = Span_{\mathbb{C}}\{L_0, W_0, z\}$. Then $\mathcal{W} = \bigoplus_{n\in\mathbb{Z}} \mathcal{W}_n$ and it is easy to check that $[\mathcal{W}_n, \mathcal{W}_m] \subset \mathcal{W}_{n+m}$, i.e. $\mathcal{W}$ is a $\mathbb{Z}$-graded Lie algebra. Furthermore, set $\mathcal{W}_- = \bigoplus_{n<0} \mathcal{W}_n, \mathcal{W}_+ = \bigoplus_{n>0} \mathcal{W}_n, \mathcal{W}_{\leq 0} = \bigoplus_{n\leq 0} \mathcal{W}_n$, then clearly $\mathcal{W}_-, \mathcal{W}_0, \mathcal{W}_+$ and $\mathcal{W}_{\leq 0}$ are subalgebras of $\mathcal{W}$.

2.2. **Partitions**.

2.2.1. We define a partition to be a non-decreasing sequence of integers $\mu = (\mu_1, \mu_2, \cdots, \mu_r), \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_r$. Denote by $\mathcal{P}$ the set of all partitions. For $\lambda = (\lambda_1, \cdots, \lambda_r) \in \mathcal{P}$, we define the length of λ to be r, denoted by $\ell(\lambda)$; if all $\lambda_i > 0, 1 \leq i \leq r$, then call λ a positive partition, and if all $\lambda_i \leq 0$ then call λ a non-positive partition. Denote by $\mathcal{P}_{\leq 0}$ the set of all non-positive partitions and by $\mathcal{P}_+$ the set of all positive partitions.

For $\lambda = (\lambda_1, \cdots, \lambda_r) \in \mathcal{P}, k \in \mathbb{Z}$, let $\lambda(k)$ denote the number of times k appears in the partition. Clearly the values $\lambda(k), k \in \mathbb{Z}$ completely determine the partition λ. So, we sometimes write λ in an alternative form, $\lambda = (\cdots(-1)^{\lambda(-1)}, 0^{\lambda(0)}, 1^{\lambda(1)}, \cdots)$. Note that $\lambda(k) = 0$ when $|k|$ is sufficiently large. Define elements $L_\lambda, W_\lambda \in \mathcal{U}(\mathcal{W})$ by

$$L_\lambda = L_{\lambda_1}L_{\lambda_2}\cdots L_{\lambda_r} = \cdots L_{-1}^{\lambda(-1)} L_0^{\lambda(0)} L_1^{\lambda(1)} \cdots$$
$$W_\lambda = W_{\lambda_1}W_{\lambda_2}\cdots W_{\lambda_r} = \cdots W_{-1}^{\lambda(-1)} W_0^{\lambda(0)} W_1^{\lambda(1)} \cdots$$

Set $\bar{0} = (\cdots(-1)^0, 0^0, 1^0, \cdots)$ and $L_{\bar{0}} = W_{\bar{0}} = 1 \in \mathcal{U}(\mathcal{W})$. We will consider $\bar{0}$ to be an element of $\mathcal{P}_{\leq 0}$ but not of $\mathcal{P}_+$. By PBW theorem, we know that $\{L_\lambda W_{\lambda'} L_\mu W_{\mu'} | \lambda, \lambda' \in \mathcal{P}_{\leq 0}, \mu, \mu' \in \mathcal{P}_+\}$ form a basis of $\mathcal{U}(\mathcal{W})$ over $S(z)$.

2.2.2. $\mathcal{U}(\mathcal{W})$ naturally inherits a grading from the one of $\mathcal{W}$. Namely, set $\mathcal{U}(\mathcal{W})_m = Span_{\mathbb{C}}\{x_1 \cdots x_k | x_i \in \mathcal{W}_{n_i}, 1 \le i \le k, \sum_{i=1}^{k} n_i = m\}$, and then $\mathcal{U}(\mathcal{W}) = \bigoplus_{m\in\mathbb{Z}} \mathcal{U}(\mathcal{W})_m$ is a $\mathbb{Z}$-graded algebra, i.e. $\mathcal{U}(\mathcal{W})_m \mathcal{U}(\mathcal{W})_n \subset \mathcal{U}(\mathcal{W})_{m+n}$. Similarly, $\mathcal{U}(\mathcal{W}_+)$ (resp. $\mathcal{U}(\mathcal{W}_{\le 0})$) inherit $\mathbb{Z}_+$-grading (resp.$\mathbb{Z}_{\le 0}$-grading) from $\mathcal{W}_+$, (resp. $\mathcal{W}_{\le 0}$). If $x \in \mathcal{U}(\mathcal{W})_m$, then we say x is a homogeneous element of degree m. If set $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_{\ell(\lambda)}$, L_λ and W_λ are homogeneous elements of degree $|\lambda|$. If $x (\neq 0)$ is not homogeneous but a sum of finitely many nonzero homogeneous elements, then denote by $mindeg(x)$ the minimum degree of its homogeneous components. Moreover, let us, for convenience, call any product of elements $L_m^t, W_n^k, z^q, (m, n \in \mathbb{Z}, t, q, k \ge 0)$, in $\mathcal{U}(\mathcal{W})$ a monomial, of height (resp. height w.r.t. L) equal to the sum of the various t's and k's (resp. t's) occurring. If $x \in \mathcal{U}(\mathcal{W})$ is a sum of monomials of height (resp. height w.r.t. L) $\le l$, we write $ht(x) \le l$ (resp. $ht_1(x) \le l$). Then we have, by PBW theorem,

Lemma. *For $m, n \in \mathbb{Z}$,let A_m stands for either L_m or W_m (resp. L_m), then $A_m^t A_n^k$ is a linear combination of $A_n^k A_m^t$ along with other monomials of height (resp. height w.r.t. L) $< t + k$.* □

2.2.3. We need some more notation. For $\lambda = (\lambda_1, \lambda_2, \cdots \lambda_r) \in \mathcal{P}, 1 \le i \le r, 0 \le j < r$,write

$$\lambda\{i\} = (\lambda_1, \cdots, \lambda_i),\ \lambda\{0\} = \bar{0},$$
$$\lambda[j] = (\lambda_{j+1}, \cdots, \lambda_r),\ \lambda[r] = \bar{0},$$
$$\lambda < i >= (\lambda_1, \cdots, \lambda_{i-1}, \hat{\lambda_i}, \lambda_{i+1}, \cdots).$$

Lemma A $ht_1([W_m, L_\lambda]) < \ell(\lambda), \forall m \in \mathbb{Z}, \lambda \in \mathcal{P}$.

Proof $[W_m, L_\lambda] = \sum\limits_{i=1}^{\ell(\lambda)} L_{\lambda\{i-1\}} [W_m, L_{\lambda_i}] L_{\lambda[i]}$. Note there is no L_k appearing in $[W_m, L_{\lambda_i}]$ and hence the lemma follows. □

Lemma B *Write $\mathcal{U}'$ for $\mathcal{U}(\mathcal{W}_+)$, $\mathcal{U}''$ for $\mathcal{U}(\mathcal{W}_{\le 0})$. Let $0 \neq x \in \mathcal{U}'_n, 0 \neq y \in \mathcal{U}''_m$ with $n > 0, m \le 0$ and $s = max\{m + n, 0\}$, then $[x, y] = \sum_{k=s}^{n} u_k$ with $u_k = \sum\limits_i y_{k,i} x_{k,i}$ where $x_{k,i} \in \mathcal{U}'_k, y_{k,i} \in \mathcal{U}''_{n+m-k}$ and $ht(y_{k,i}) < ht(y)$ if $k = n, y_{k,i} \neq 0$. Further, if assume $x \in (\mathcal{W}_+)_n$, then $x_{k,i} \in (\mathcal{W}_+)_k$ and $ht(y_{k,i}) < ht(y)$ whenever $k > 0, y_{k,i} \neq 0$.*

Proof By direct checking. □

2.3 **Whittaker Module**. Observe that $\mathcal{W}_+$ is a subalgebra of $\mathcal{W}$ generated by L_1, L_2, W_1, W_2 As in [12], we define Whittaker modules for $\mathcal{W}$ as what follows.

2.3.1. **Definition**. One says a Lie algebra homomorphism $\varphi : \mathcal{W}_+ \to \mathbb{C}$ non-singular, if $\varphi(L_i)\varphi(W_i) \neq 0$, $i = 1, 2$. For a $\mathcal{W}$-module V, a vector $v \in V$ is said to be a Whittaker vector if $xv = \varphi(x)v$ for all $x \in \mathcal{W}_+$. Furthermore, if v generates V, we call V a Whittaker module of type φ (φ is required to be non-singular in this case) and v a cyclic Whittaker vector of V.

2.3.2. For a given non-singular Lie algebra homomorphism $\varphi : \mathcal{W}_+ \to \mathbb{C}$, define $\mathbb{C}_\varphi$ to be

the one-dimensional $\mathcal{W}_+$-module given by the action $x\alpha = \varphi(x)\alpha$ for all $x \in \mathcal{W}_+$ and $\alpha \in \mathbb{C}$. Then the induced $\mathcal{W}$-module

$$M_\varphi = \mathcal{U}(\mathcal{W}) \otimes_{\mathcal{U}(\mathcal{W}_+)} \mathbb{C}_\varphi,$$

is a Whittaker module of type φ with a cyclic Whittaker vector $w = \mathbf{1} \otimes 1$. By PBW theorem, it's easy to see that $\{z^k L_\lambda W_\mu w \mid \lambda, \mu \in \mathcal{P}_{\leq 0}, k \geq 0\}$ is a basis of M_φ as a vector space over $\mathbb{C}$.

Besides, for any $\xi \in \mathbb{C}$, obviously $(z-\xi)M_\varphi$ is a submodule of M_φ. Define $L_{\varphi,\xi} = M_\varphi/(z-\xi)M_\varphi$ and denote the canonical homomorphism by p_ξ. Then $L_{\varphi,\xi}$ is a Whittaker module for $\mathcal{W}$. The following lemma makes M_φ become a universal Whittaker module.

Lemma *Fix φ and M_φ as above. Let V be a Whittaker module of type φ generated by a Whittaker vector w_v. The there is a unique map $\phi : M_\varphi \to V$ taking $w = 1 \otimes 1$ to w_v.*

Proof. Uniqueness is obvious. For $u \in \mathcal{U}(\mathcal{W})$, one can write, by PBW,

$$u = \sum_\alpha b_\alpha n_\alpha, b_\alpha \in \mathcal{U}(\mathcal{W}_{\leq 0}),\ n_\alpha \in U(W_+).$$

If $uw = 0$ then $uw = \sum\limits_\alpha b_\alpha \psi(n_\alpha) w = 0$. Therefore, $\sum\limits_\alpha b_\alpha \psi(n_\alpha) = 0$. Now it's easy to see that the map $\phi : M_\varphi \to V$, defined by $\phi(uw) = uw_v$, is well defined. $\square$

2.3.3. Write M for M_φ, $\mathcal{U}'$ for $\mathcal{U}(\mathcal{W}_+)$, and $\mathcal{U}''$ for $\mathcal{U}(\mathcal{W}_{\leq 0})$. For $k \leq 0$, define $M_k = \{xw \mid x \in \mathcal{U}''_k\}$ and then clearly $M = M_0 \oplus M_{-1} \oplus M_{-2} \oplus \cdots$. We say that a nonzero homogeneous vector v in M is of degree k if $v \in M_k$. If $v(\neq 0)$ is not homogeneous but a sum of finitely many nonzero homogeneous vectors, then define $mindeg(v)$ to be the minimum degree of its homogeneous components. Meanwhile, for any nonzero vector $v \in M$, let $d = mindeg(v)$ and then there uniquely exist $v_i \in \mathcal{U}''_i, 0 \geq i \geq d$ such that $v = \sum\limits_{i=d}^{0} v_i w$ with $v_d \neq 0$. Then define $\ell(v) = ht(v_d)$ and $\ell'(v) = ht_1(v_d)$.

§3. Whittaker Vectors in M_φ and $L_{\varphi,\xi}$

In this section, we characterize the Whittaker vectors in M_φ and $L_{\varphi,\xi}$. Fix a nonsingular Lie algebra homomorphism $\varphi : \mathcal{W}_+ \to \mathbb{C}$, and let $w = \mathbf{1} \otimes 1 \in M_\varphi$.

3.1. **Proposition** *If $w' \in M_\varphi$ is a Whittaker vector (of type φ), then $w' = p(z)w$ for some $p(z) \in S(z)$.*

Proof. Write $w' = \sum\limits_{\lambda,\mu \in \mathcal{P}_{\leq 0}} p_{\lambda,\mu}(z) L_\lambda W_\mu w$. Clearly it's enough to show that $p_{\lambda,\mu} = 0$ unless $\lambda = \bar{0}$ and $\mu = \bar{0}$.

a). Suppose there, at least, exists a $\lambda \neq \bar{0}$ such that $p_{\lambda,\mu}(z) \neq 0$ for some μ. Now let $A = \{\lambda \mid p_{\lambda,\mu} \neq 0, \lambda \in \mathcal{P}_{\leq 0}\}$, $l = max\{\ell(\lambda) \mid \lambda \in A\}$, and $B = \{\lambda | \ell(\lambda) = l, \lambda \in A\}$. Put

$m_0 = min\{\lambda_1 \,|\, \lambda \in B\}$ and $B' = \{\lambda \in B \,|\, \lambda_1 = m_0\}$. Note that if $\lambda \in B, \lambda_i = m_0$ for some i, then $\lambda_1 = \cdots = \lambda_i = m_0$ and hence $\lambda \in B'$, $\lambda < i >= \lambda < 1 >$.

Let $m = 2 - m_0 (\geq 2)$ and then $\forall \lambda \in B, m + \lambda_i > 2$ unless $\lambda_i = m_0$. Consider

$$\begin{aligned}(W_m - \varphi(W_m))w' &= \sum_{\lambda,\mu} p_{\lambda,\mu}(z)[W_m, L_\lambda W_\mu]w \\ &= \sum_{\lambda,\mu} p_{\lambda,\mu}(z)[W_m, L_\lambda]W_\mu w = D_1 + D_2.\end{aligned}$$

where $D_1 = \sum\limits_{\mu,\lambda \in B} p_{\lambda,\mu}(z)[W_m, L_\lambda]W_\mu w$, and $D_2 = \sum\limits_{\mu,\lambda \notin B} p_{\lambda,\mu}(z)[W_m, L_\lambda]W_\mu w$. Then, clearly

$$\ell'(p_{\lambda,\mu}(z)[W_m, L_\lambda]W_\mu w) \leq l - 2,$$

for any $\lambda \notin B$ and hence $\ell'(D_2) \leq l - 2$, by 2.2.3 Lemma A. Meanwhile,

$$\begin{aligned}D_1 &= \sum_{\mu,\lambda \in B} p_{\lambda,\mu}(z) \sum_{i=1}^{\ell(\lambda)} L_{\lambda\{i-1\}}[W_m, L_{\lambda_i}]L_{\lambda[i]}W_\mu w \\ &= \sum_{\mu,\lambda \in B} p_{\lambda,\mu}(z) \sum_{i=1}^{\ell(\lambda)} L_{\lambda\{i-1\}}(\lambda_i - m)W_{m+\lambda_i}L_{\lambda[i]}W_\mu w \\ &= D_1' + D_1'',\end{aligned}$$

where

$$D_1' = \sum_{\mu,\lambda \in B} p_{\lambda,\mu}(z) \sum_{i=1}^{\ell(\lambda)} L_{\lambda<i>}(\lambda_i - m)W_{m+\lambda_i}W_\mu w,$$

$$D_1'' = \sum_{\mu,\lambda \in B} p_{\lambda,\mu}(z) \sum_{i=1}^{\ell(\lambda)} L_{\lambda\{i-1\}}(\lambda_i - m)[W_{m+\lambda_i}, L_{\lambda[i]}]W_\mu w.$$

Then, we have $\ell'(D_1'') \leq l - 2$, by lemma A. But, since $W_{m+\lambda_i}w = 0$ when $m + \lambda_i > 2$,

$$\begin{aligned}D_1' &= \sum_{\mu,\lambda \in B'} p_{\lambda,\mu}(z) \sum_{i=1}^{\lambda(m_0)} (2m_0 - 2)L_{\lambda<i>}W_2 W_\mu w \\ &= \sum_{\mu,\lambda \in B'} (2m_0 - 2)\varphi(W_2)\lambda(m_0)p_{\lambda,\mu}(z)L_{\lambda<1>}W_\mu w.\end{aligned}$$

So, D_1' equals to a linear combination of some vectors v's in M_φ with $ht_1(v) = l-1$ which are linearly independent from each other. Therefore D_1' and $D_1'' + D_2$ are linearly independent. Thus, $(W_m - \varphi(W_m))w' = D_1' + D_1'' + D_2 \neq 0$ which contradicts with the hypothesis that w' is a Whittaker vector.

b). Suppose $\exists \mu \neq \bar{0}, p_{\bar{0},\mu}(z) \neq 0$, and $p_{\lambda,\mu}(z) = 0, \forall \lambda \neq \bar{0}$. Then we write $w' = \sum\limits_{\mu \in \mathcal{P}_{\leq 0}} p_{\bar{0},\mu}(z)W_\mu w$. Let $C = \{\mu \,|\, p_{\bar{0},\mu}(z) \neq 0\}, n_0 = min\{\mu_1 | \mu \in C\}$, and $C' = \{\mu \,|\, \mu_1 =$

$n_0, \mu \in C\}$. And if, $\forall \mu \in C, \mu_i = n_0$ for some i, then clearly $\mu_1 = \cdots = \mu_i = n_0$ and $\mu \in C'$. Now let $m = 2 - n_0$ and then $m + \mu_i > 2$ unless $\mu_i = n_0$.

Consider

$$\begin{aligned}(L_m - \varphi(L_m))w' &= \sum_\mu p_{\bar{0},\mu}(z)[L_m, W_\mu]w \\ &= \sum_\mu p_{\bar{0},\mu}(z) \sum_{i=1}^{\ell(\mu)} W_{\{i-1\}}[L_m, W_{\mu_i}]W_{\mu[i]}w \\ &= \sum_\mu p_{\bar{0},\mu}(z) \sum_{i=1}^{\ell(\mu)} (m - \mu_i) W_{\mu<i>} W_{m+\mu_i}.\end{aligned}$$

Since $W_{m+\mu_i} w = 0$ if $m + \mu_i > 2$, we have

$$(L_m - \varphi(L_m))w' = \sum_{\mu \in C'} p_{\bar{0},\mu}(z)\mu(n_0)(m - n_0)\varphi(W_2)W_{\mu<1>}w.$$

But this is a sum of terms that are linearly independent from each other and hence $(L_m - \varphi(L_m))w' \neq 0$ which contradicts with the hypothesis that w' is a Whittaker vector. $\square$

3.2. **Proposition** *Let $w = \mathbf{1} \otimes 1 \in M_\varphi$ and $\overline{w} = \overline{\mathbf{1} \otimes 1} \in L_{\varphi,\xi}$. If $w' \in L_{\varphi,\xi}$ is a Whittaker vector then $w' = c\overline{w}$ for some $c \in \mathbb{C}$.*

Proof $L_{\varphi,\xi}$ has a basis in the form $\{L_\lambda W_\mu \overline{w} \mid \lambda, \mu \in \mathcal{P}_{\leq 0}\}$. In fact, clearly it is enough to show this set is linearly independent. Suppose there are $a_{\lambda,\mu} \in \mathbb{C}$, with at most finitely many $a_{\lambda,\mu} \neq 0$, such that

$$0 = \sum_{\lambda,\mu \in \mathcal{P}_{\leq 0}} a_{\lambda,\mu} L_\lambda W_\mu \overline{w} = \overline{\sum_{\alpha,\mu} a_{\lambda,\mu} L_\lambda W_\mu w}$$

in $L_{\varphi,\xi}$. So

$$\sum_\mu a_{\lambda,\mu} L_\lambda W_\mu w = (z - \xi) \sum_{\lambda,\mu} \sum_{i=1}^{k} b_{\lambda,\mu,i} z^i L_\lambda W_\mu w$$

for some $k > 0$ and $b_{\lambda,\mu,i} \in \mathbb{C}$. The right hand side of this expression can be rewritten as a linear combination of $\mathbb{C}$-basis vectors $z^i L_\lambda W_\mu w$ in $M_{\varphi,\xi}$. Then comparing all the coefficients of the two sides of the above equation, we can deduce that $a_{\lambda,\mu} = 0$, for all λ, μ.

With this fact now established, the same argument as in Proposition 3.1 works well here, if we simply replace the polynomials $p_{\lambda,\mu}(z)$ in z with scalars $p_{\lambda,\mu}$. $\square$

3.3. **Lemma** *Let V be a submodule of M_φ. Then V contains a nonzero Whittaker vector.*

Proof Suppose V contains no nonzero Whittaker vectors. Use the notation as in 2.3.3.

Now let $n_0 = max\{mindeg(v) | v \neq 0, v \in V\}$, $l = min\{\ell(v) | mindeg(v) = n_0, v \in V\}$. Take a $u \in V$ such that $mindeg(u) = n_0$, $\ell(u) = l$. Write $u = u_0 w + u_{-1} w + \cdots + u_{n_0} w, u_i \in \mathcal{U}_i'', u_{n_0} \neq 0$. Since u is not a Whittaker vector, there exists a $x \in (\mathcal{W}_+)_m$, for some $m > 0$

such that $u' := xu - \varphi(x)u = \sum_{i=n_0}^{0} [x, u_i]w \neq 0$. Note u' is contained in V. Clearly, by 2.2.3 Lemma B, $mindeg([x, u_i]w) \geq i \geq n_0$, if $[x, u_i] \neq 0$. So we have $mindeg(u') \geq n_0$ and hence $mindeg(u') = n_0$ for the definition of n_0. In this case, $[x, u_{n_0}]w \neq 0$ and $mindeg([x, u_{n_0}]) = n_0$. But this forces $\ell([x, u_{n_0}]w) < ht(u_{n_0}) = \ell(u)$ by Lemma B. Thus, $\ell(u') < l$, which contradicts with the definition of l. $\square$

§4. Simple Whittaker Modules

Now we are ready to determine all the simple Whittaker modules. $\mathcal{W}, \varphi : \mathcal{W}_+ \to \mathbb{C}, M = M_\varphi, w = \mathbf{1} \otimes 1$, as above.

4.1 **propostion** *Any nontrivial submodule of a Whittaker module of type φ contains a nontrivial Whittaker submodule of type φ.*

Proof Let V be a Whittaker module of type φ. We first show it for the case $V = M/IM$ where I is an ideal of $A = S(z)$. Note that $V = M/IM$ admits a basis, $\{L_\lambda W_\mu \bar{w} \,|\, \lambda, \mu \in \mathcal{P}_{\leq 0}\}$ over A/I. Namely, note that $M = \bigoplus_{\lambda,\mu \in \mathcal{P}_{\leq 0}} AL_\lambda w$. Hence,

$$M/IM = A/I \otimes_A M = A/I \otimes_A (\bigoplus_{\lambda,\mu \in \mathcal{P}_{\leq 0}} Ax_\lambda w) = \bigoplus_{\lambda,\mu \in \mathcal{P}_{\leq 0}} (A/I)x_\lambda \bar{w}.$$

Then applying the argument in the proof of Lemma 3.3, one sees immediately that the proposition holds in this case.

Note that with the fact that $V = M/IM$ admits a basis, $\{L_\lambda W_\mu \bar{w} \,|\, \lambda, \mu \in \mathcal{P}_{\leq 0}\}$, over A/I, one sees that Proposition 3.1 holds for $V = M/IM$. Obviously it is enough to show that there are no other cases. Now, the proposition follows immediately from the claim below.

Claim: *Let N be a submodule of $M = M_\varphi$. Then $N = IM$ for some $I \subseteq A = S(z)$.*

Proof of the claim: Set $I = \{x \in A \,|\, xw \in N\}$. One immediately sees that I is an ideal of A and $IM \subseteq N$. So we can view N/IM as a submodule of M/IM. If $N \neq IM$, then there exists $p\bar{w} \in N/IM$, with $p\bar{w} \neq 0, p \in A$, since 3.1 and 3.3 hold for M/IM. So $pw \in N$ and hence $p \in I$. Therefore $pw \in IM$, which contradicts with the fact that $p\bar{w} \neq 0$ in N/IM. Thus, $N = IM$. $\square$

4.2. **Theorem** *For any $\xi \in \mathbb{C}$, $L_{\varphi,\xi}$ is simple and any simple Whittaker module of type φ is of form $L_{\varphi,\xi}$.*

Proof The fist statement follows from Proposition 3.2 and 4.1. For the second one, let V be a simple Whittake module of type φ. Consider a surjection

$$\pi : M \to V.$$

By Schur's Lemma, there exists a $\xi \in \mathbb{C}$ such that $zv = \xi v, \forall v \in V$. Hence, $\pi((z-\xi)M_\varphi) = 0$ i.e. π factor through $L_{\varphi,\xi}$ and hence $V \simeq L_{\varphi,\xi}$. $\square$

4.3. We develop two more results to close this section.

4.3.1. **Lemma** *Set* $\mathcal{U}=\mathcal{U}(\mathcal{W}),\ L=\mathcal{U}(z-\xi\cdot 1)+\sum_{\lambda,\mu\in\mathcal{P}_+}\mathcal{U}(L_\lambda W_\mu-\varphi(L_\lambda W_\mu)),$ *and* $V=\mathcal{U}/L$. *Then* $V\simeq L_{\varphi,\xi}$.

Proof Note that $\bar{1}$ in V is obviously a Whittaker vector of type φ, and also z acts on V by some scalar ξ. By the universal property of M_φ, we have a surjection

$$\psi: M_\varphi \to V,$$

sending w to $\bar{1}$. But then $\psi((z-\xi)M)=(z-\xi)V=0$. Hence

$$(z-\xi)M_\varphi \subseteq \ker\psi \subsetneq M_\varphi,$$

and therefore, $(z-\xi)M_\varphi=\ker\varphi$, i.e. $V\simeq L_{\varphi,\xi}$. □

4.3.2. **Proposition** *Suppose V is a Whittaker module, and z acts on V by the scalar $\xi\in\mathbb{C}$. Then V is isomorphic to $L_{\varphi,\xi}$ and therefore, if w is a cyclic Whittaker vector for V,* $Ann_{\mathcal{U}(\mathcal{W})}(w)=\mathcal{U}(\mathcal{W})(z-\xi\cdot 1)+\sum_{\lambda,\mu\in\mathcal{P}_+}\mathcal{U}(\mathcal{W})(L_\lambda W_\mu-\varphi(L_\lambda W_\mu)).$

Proof Let $\pi:\mathcal{U}(\mathcal{W})\to V$, with $\pi(1)=w$ be the canonical homomorphism, and $K=\mathrm{Ker}\pi$. Then $K\subsetneq\mathcal{U}(\mathcal{W})$ and

$$L:=\mathcal{U}(\mathcal{W})(z-\xi\cdot 1)+\sum_{\lambda,\mu}\mathcal{U}(\mathcal{W})(L_\lambda W_\lambda-\varphi(L_\lambda W_\lambda))\subset K.$$

By 4.3.1, L is maximal and thus $K=L$ and $V\simeq\mathcal{U}(\mathcal{W})/L\simeq L_{\varphi,\xi}$. □

§5. Submodules Of Whittaker Modules

We now characterize arbitrary Whittaker modules, with generating Whittaker vector w, in terms of the annihilator $\mathrm{Ann}_{S(z)}(w)$. The results and their proofs here are essentially same as [12].

5.1. **Decomposition of Whittaker Modules**.

5.1.1. **Lemma** *Suppose that V is a Whittaker module of type φ with cyclic Whittaker vector w and assume that $Ann_{S(z)}(w)=(z-\xi\cdot 1)^a$ for some $a>0$. define $V_i\triangleq\mathcal{U}(\mathcal{W})(z-\xi\cdot 1)^i w,\ \ 0\le i\le a$. Then*

1) $V=V_0\supseteq V_1\supseteq\cdots\supseteq V_a=0$ form a composition series with $V_i/V_{i+1}\simeq L_{\varphi,\xi}$;

2) $V_0\cdots V_a$ are all the submodules of V.

Proof 1) Clearly $V=V_0\supseteq V_1\supseteq\cdots\supseteq V_a=0$. Since z acts by the scalar ξ on V_i/V_{i+1}, it follows by Proposition 4.3.2 that V_i/V_{i+1} is simple and isomorphic to $L_{\varphi,\xi}$.

2) If M is any maximal submodule of V, then V/M is simple and it's easy to use Proposition 4.3.2 to deduce that $V/M\simeq L_{\varphi,\xi}$. So $(z-\xi\cdot 1)V\subset M$, i.e. $V_1\subset M$. Therefore

$V_1 = M$. A similar argument shows that V_{i+1} is the unique maximal submodule of $V_i, \forall i < a$. $\square$

5.1.2. **Theorem** *Suppose that V is a Whittaker module of type φ with cyclic Whittaker vector w, and $Ann_{S(z)}(w) \neq 0$. Let $p(z)$ is the unique monic generator of $Ann_{S(z)}(w)$. Write $p(z) = \prod_{i=1}^{k}(z - \xi_i \cdot 1)^{a_i}$, $\xi_i \neq \xi_j$, $i \neq j$, and $w_j = p_j(z)w$, where $p_j(z) = \prod_{i \neq j}(z - \xi_i \cdot 1)^{a_i}$. Then $V_j := \mathcal{U}(\mathcal{W})w_j$ is a Whittaker module of type φ with cyclic Whittaker vector w_i and $V = V_1 \oplus \cdots \oplus V_k$. Furthermore, the submodules $V_1, \cdots, V_k$ are indecomposable and the composition length of V_j is a_j.*

Proof Since $\gcd(p_1(z), \cdots, p_k(z)) = 1$, there exist polynomails $q_i(z), 1 \leq i \leq k$ such that $\sum q_i(z)p_i(z) = 1$. Therefore $1 \cdot w = \sum q_i(z)p_i(z)w \in V_1 + \cdots + V_k$ and thus, $V = V_1 + \cdots + V_k$. To show that the sum is direct, first note that $p_j(z)w_i = 0, i \neq j$. Hence

$$w_i = 1 \cdot w_i = q_i(z)p_i(z)w_i.$$

Now if $u_1w_1 + \cdots + u_kw_k = 0$, then

$$0 = q_i(z)p_i(z)(\sum_j u_jw_j) = u_iq_i(z)p_i(z)w_i = u_iw_i.$$

and this implies that the sum is direct. The rest follows from 3.3.3 since $\mathrm{Ann}_{s(z)}(w_i) = (z - \xi_i)^{a_i}$.

Remark. It's easy to see, from the proof above, that 1) any submodule of V is the direct sum of its intersections with the V_j's; 2) $(z - \xi_i)V_j = V_j, i \neq j$.

5.1.3. **Corollary** *Suppose that V is a Whittaker module of type φ with cyclic Whittaker vector w, and $Ann_{S(z)}(w) = (p(z))$, where $p(z)$ is a monic polynomial. Then $Ann_{\mathcal{U}(\mathcal{W})}(w) = \mathcal{U}(\mathcal{W})p(z) + \sum_{\lambda,\mu}\mathcal{U}(\mathcal{W})(L_\lambda W_\mu - \varphi(L_\lambda W_\mu))$.*

Proof. By induction on $deg(p(z))$. Assume $deg(p) > 1$. Write $p(z) = (z - \xi \cdot 1)p'(z), p'(z) \neq 1$, and $p'(x)$ is monic. Then $(z - \xi \cdot 1)w \neq 0$. Let $V' = \mathcal{U}(\mathcal{W})w'$ and $w' = (z - \xi \cdot 1)w$. Then obviously V' is a Whittaker module, and $\mathrm{Ann}_{S(z)}w' = S(z)p'(z)$. By induction, we have $\mathrm{Ann}_{\mathcal{U}(\mathcal{W})}(w') = \mathcal{U}(\mathcal{W})p'(z) + \sum_{\lambda,\mu}\mathcal{U}(\mathcal{W})(L_\lambda W_\mu - \varphi(L_\lambda W_\mu)$. Observe that $\mathrm{Ann}_{S(z),V/V'}(\overline{w}) = S(z)(z - \xi \cdot 1)$, where $\overline{w} = w + V'$. Since $\mathrm{Ann}_{\mathcal{U}(\mathcal{W})}(w) \subset \mathrm{Ann}_{\mathcal{U}(\mathcal{W})}(\overline{w})$, So for any $u \in \mathrm{Ann}_{\mathcal{U}(\mathcal{W})}(w)$, there exist $u_0, u_{\lambda,\mu} \in \mathcal{U}(\mathcal{W})$ such that

$$u = u_0(z - \xi \cdot 1) + \sum_{\lambda,\mu \in \mathcal{P}_+} u_{\lambda,\mu}(L_\lambda W_\mu - \varphi(L_\lambda W_\mu).$$

But $\sum_{\lambda,\mu \in \mathcal{P}_+} u_{\lambda,\mu}(L_\lambda W_\mu - \varphi(L_\lambda W_\mu) \in \mathrm{Ann}_{\mathcal{U}(\mathcal{W})}(w)$. So $u_0(z - \xi \cdot 1) \in \mathrm{Ann}_{\mathcal{U}(\mathcal{W})}(w)$. Thus, $0 = u_0(z - \xi \cdot 1)w = u_0w'$, and hence $u_0 \in \mathrm{Ann}_{\mathcal{U}(\mathcal{W})}(w') = \mathcal{U}(\mathcal{W})p'(z) + \sum_{\lambda,\mu}\mathcal{U}(\mathcal{W})(L_\lambda W_\mu - \varphi(L_\lambda W_\mu)$.

This implies that $u \in \mathcal{U}(\mathcal{W})p(z) + \sum_{\lambda,\mu}\mathcal{U}(\mathcal{W})(L_\lambda W_\mu - \varphi(L_\lambda W_\mu)$. $\square$

5.2. **Submodules of** M_φ.

5.2.1. **Lemma** *Suppose that* V *is a Whittaker module of type* φ *with cyclic Whittaker vector* w*. If* $Ann_{S(z)}(w) = 0$*, then* $V \simeq M_\varphi$*.*

Proof Consider a surjection, $\pi : M_\varphi \to V$. Let $K = \mathrm{Ker}\pi$. If $K \neq 0$, then by Lemma 3.3, there exists $w' \in K$ so that w' is a nonzero Whittaker vector of type φ. Hence by 3.1, we have $w' = p(z)(\mathbf{1} \otimes 1)$ for some nonzero polynomial $p(z)$, and then $0 = \pi(p(z)(\mathbf{1} \otimes 1)) = p(z)w$. Therefore, $p(z) \in \mathrm{Ann}_{S(z)}w = 0$. Impossible. So, $K = 0$. □

5.2.2 **Theorem** *Set* $M = M_\varphi, w = \mathbf{1} \otimes 1$*. Let* V *be a submodule of* M*. Then* $V \simeq M$*. Furthermore,* V *is generated by a Whittaker vector of the form* $q(z)w$ *for some polynomial* $q(z)$*.*

Proof If $V = M$, then done. Now assume $V \neq M$. Note first that $\mathcal{U}(\mathcal{W})f(z)w$ is a Whittaker module with cyclic Whittaker vector $f(z)w$, for any nonzero polynomial $f(z)$. Thus $\mathcal{U}(\mathcal{W})f(z)w \simeq M$ by Lemma 5.2.1, since $\mathrm{Ann}_{S(z)}(w) = 0$. Now Proposition 3.1 and Lemma 3.3 imply that there exist a nonzero polynomial $p(z)$ such that $w' := p(z)w$ is a Whittaker vector contained in V. Therefore

$$M' := \mathcal{U}(\mathcal{W})p(z)w \simeq M.$$

We operate on the triple $(M' \subseteq V \subseteq M)$. Write $p(z) = \prod_{i=1}^r (z - \xi_i)^{a_i}, a_i > 0, \xi_i \neq \xi_j$. Denote by ψ the canonical map $M \to M/M'$, and then $V = \psi^{-1}(\psi(V))$ since $V \supset M'$. Applying Theorem 5.1.2 to M/M', we have that

$$M/M' = \overline{M_1} \oplus \cdots \oplus \overline{M_r}$$

where $\overline{M_i} = \mathcal{U}(\mathcal{W})\overline{w_i}$,and $\overline{w_i} = \prod_{j\neq i}(z - \xi_j \cdot 1)^{a_j}\overline{w}$, with $\overline{w} = \psi(w)$. By Remark 5.1.2, $\psi(V) = \oplus_i(\psi(V) \cap \bar{M}_i)$. Then, without any loss, we may assume $\psi(V) \cap \overline{M_1} \neq \overline{M_1}$. By Lemma 5.1.1, $(\psi(V) \cap \overline{M_1}) \subseteq (z - \xi_1 \cdot 1)\overline{M_1}$. Let $N = (z - \xi_1 \cdot 1)\overline{M_1} + \overline{M_2} + \cdots + \overline{M_r}$ and hence $\psi(V) \subseteq N$. Using Remark 5.1.2, we can deduce that $N = (z - \xi_1)M/M'$. Therefore $M' \subseteq V \subseteq \psi^{-1}(N)$. However $\psi^{-1}(N) = (z - \xi_1)M \simeq M$. With this isomorphism, we obtain another triple $(M'' \subseteq V' \subseteq M)$ with $M'' \simeq M', V' \simeq V$. Observe that the composition length of M/M'' is one less than the one of M/M'. Therefore, to complete the proof, we only need to repeat the above operation on the new triple. □

Remark. One may also use the facts developed in the proof of Proposition 4.1 to give another proof.

5.3. Now we can characterize submodules of any Whittaker modules.

5.3.1. **Lemma** *Suppose that* V *is an indecomposable Whittaker module of type* φ *with cyclic Whittaker vector* w*, then* $Wh(V) = S(z)w$*. where* $Wh(V)$ *stands for the set of all Whittaker vectors (including* 0*) of* V*.*

Proof a) $\mathrm{Ann}_{S(z)}(w) = 0$, the result follows from Lemma 5.2.1 and Proposition 3.1.

b) $\mathrm{Ann}_{S(z)}(w) \neq 0$. Use induction on $\ell(V)$, the composition length of V. By 5.1, we have

$$V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_a = 0.$$

where V_i is generated by the cyclic Whittaker vector $w_i = (z - \xi \cdot 1)^i w$ for some $\xi \in \mathbb{C}$, and $a = \ell(V)$. For $w' \in Wh(V)$, we conclude, by 5.1.1 and 3.2, that $\overline{w'} = c\overline{w}$ in V/V_1 for some $c \in \mathbb{C}$. Therefore $w' = cw + w''$, with $w'' \in V_1$. Note $w'' = w' - cw$ is also a Whittaker vector. Now $\ell(V_1) = a - 1$ and then by induction, $w'' = p(z)w_1 = p(z)(z - \xi \cdot 1)w$ for some $p(z) \in S(z)$, therefore $w' = cw + p(z)(z - \xi \cdot 1)w$. So $Wh(V) = S(z)w$. □

5.3.2 **Corollary** *Suppose that V is a Whittaker module of type φ. Then 1) there exist a series of indecomposable Whittaker modules $V_i, 1 \leq i \leq r$, such that $V = \oplus_{i=1}^r V_i$; 2) Suppose w is a cyclic Whittaker vector of V. Then any submodule of V is a Whittaker module generated by a vector in $Wh(V) = S(z)w$.*

Proof 1) follows from Theorem 5.1.2 and Lemma 5.2.1. 2) If $V \simeq M_\varphi$,then it follows from Theorem 5.2.2; if not, it follows from 5.1 and 5.3.1. □

References

[1] D. Arnal and G. Pinczon, On algebraically irreducible representations of he Lie algebra $\mathfrak{sl}_2$ *J. Math. Phys.* **15** (1974), 350–359.

[2] G. Benkart and M. Ondrus, Whittaker modules for Generalized Weyl Algebras, arXiv:0803.3570.

[3] R. Block, The irreducible representations of the Lie algebra $\mathfrak{sl}_2$ and of the weyl algebra, *Adv. Math.* **39** (1981), 69–110.

[4] K. Christodoulopoulou, Whittaker modules for Heisenberg algebras and imaginary Whittaker modules for affine Lie algebras, *J. Alg.* **320** (2008) 2871–2890.

[5] H. Chen and J. Li, Left-symmetric Algebra Structures on the W Algebra $W(2,2)$, preprint.

[6] S. Gao, C. Jiang, Y. Pei, The derivations, central extensions and automorphisms of the Lie algebra the W, preprint, arXiv:0801.3911v1.

[7] B. Kostant, On Whittaker vectors and representation theory, *Invent. Math.*, **48** (1978), 101–184.

[8] D. Liu, S. Gao, L. Zhu, Classification of irreducible weight modules over the W-algebra $W(2,2)$, preprint, arXiv:0801.2603v2.

[9] J. Li, Y. Su, Lie bialgebra structures on the W-algebra $W(2,2)$, arXiv:0801.4144v1.

[10] D. Liu, L. Zhu, Classification of Harish-Chandra over the W-algebra $W(2,2)$, preprint, arXiv:0801.2601v2.

[11] M. Ondrus, Whittaker modules for $\mathcal{U}_q(\mathfrak{sl}_2)$, *J. Alg.* **289** (2005), 192–213.

[12] M. Ondrus and E. Wiesenr, Whittaker modules for the Virasoro algebra, arXiv:0805.2686.

[13] A. Sevostyanov, Quantum deformation of Whittaker modules and Toda lattice, *Duke Math. J.*, (2000), **204** 211–238.

[14] W. Zhang, C. Dong, W-algebra $W(2,2)$ and the Vertex operator algebra $L(\frac{1}{2},0)\otimes L(0,\frac{1}{2})$, preprint, arXiv:0711.4624v1.

[15] X. Zhang and S. Tan, Whittaker modules and a class of new modules similar as Whittaker modules for the Schrödinger-Virasoro algebra, arXiv:0812.3245v1.

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